International Journal of Theoretical Physics, Vol. 37, No. 12, 1998

Lorentz-Invariant Pseudo-Differential Wave Equations

D. G. Barci,[1] C. G. Bollini,[2] L. E. Oxman,[3] and M. C. Rocca[2]

Received March 17, 1998

We define in a consistent way nonlocal pseudo-differential operators acting on a space of analytic functionals. We discuss the relation of our method to other definitions for nonlocal operators. We show how to solve homogeneous and inhomogeneous equations associated with nonlocal pseudo-differential operators, and we extend the formalism to d-dimensional space-time solving, in particular, the fractional wave and Klein–Gordon equations. We also study in this context the nonlocal equations obtained from effective QED when integrating over the fermionic degrees of freedom.

1. INTRODUCTION

Interest in nonlocal field theories has always been present in theoretical physics, associated with several different motivations. Wheeler and Feynman [1] considered a description of the interaction between charged particles where the electromagnetic field does not appear as a dynamical variable (action at a distance).

More recently, efforts have been made to use nonlocal theories in connection with the understanding of quark confinement and anomalies [2, 3] and in string theories containing nonlocal vertices [4, 5].

Besides the possibility of nonlocal interactions, a field theory can also display nonlocal kinetic terms.

Before renormalization theory became well established, the possibility was considered of formulating finite theories by means of nonlocal kinetic

[1] Universidade do Estado do Rio de Janeiro, Instituto de Física, Departamento de Física Teórica, Rio de Janeiro, Brazil; e-mail: barci@vmesa.uerj.br.
[2] Universidad Nacional de La Plata, C.C. 67 (1900), La Plata, Argentina; e-mail: rocca@venus.fisica.unlp.edu.ar.
[3] Departamento de Física, Pontifícia Universidade Católica, 22452-970, Rio de Janeiro Brazil; e-mail: oxman@fis.puc-rio.br.

0020-7748/98/1200-3015$15.00/0

Lagrangians. Pais and Uhlembeck [6] were one of the first to analyze nonlocal theories in this context.

The analytic regularization method introduced in ref. 7 can be thought of as associated to nonlocal kinetic terms in the Lagrangian, specifically to fractional wave and Klein–Gordon equations.

More recently, this type of nonlocality has arisen in effective field theories when integrating over some degrees of freedom in an underlying local field theory [8–10] and in the context of bosonization in three dimensions [11–14].

At a classical level, nonlocal equations containing arbitrary powers of the D'Alambertian (;□) have been studied in ref. 15. In this reference a nontrivial relation between the number of dimensions and the power of the operator was established in order to satisfy the Huygens principle [16]. In particular, in 2 + 1 dimensions, the usual wave equation leads to a Green function that does not satisfy Huygens' principle, while a nonlocal equation with $\Box^{1/2}$ does satisfy this principle (see also ref. 17). It is not by chance that the pseudo-differential operator $\Box^{1/2}$ also appears in the process of bozonization in 2 + 1 dimensions. In refs. 11 and 12 a mapping was established between Dirac's kinetic term and bosonic terms involving $\Box^{1/2}$. In ref. 18 it is shown how similar terms appear when (3 + 1)-dimensional QED is projected to a physical plane. The kinetic term obtained $F_{\mu\nu}\Box^{-1/2}F^{\mu\nu}$, reproduces correctly the r^{-1} Coulomb potential instead of the usual logarithmic behavior of 2 + 1 QED. This fact was first noticed in ref. 16.

Also, in ref. 19, a fractional generalized Fokker–Planck–Kolmogorov equation is proposed to describe anomalous transport in Hamiltonian systems, and in refs. 20 and 21 some particular Green functions for fractional diffusion and fractional wave equations were obtained.

In all these references, under justified physical assumptions, the nonlocal kinetic operator definition includes an *a priori* prescription in a way that parallels the fractional derivative defined in ref. 22. In that reference, the prescription used for the fractional operator leads to a Green function having a prescription which is uniquely determined, that is, the usual freedom to choose the Green functions for a given differential operator is absent in that approach. In other words, using that approach, the nonlocal homogeneous equations have no nontrivial solutions.

The problem of defining nonlocal pseudo-differential operators having nontrivial solutions to the associated homogeneous equations has not been considered in the traditional literature.

In refs. 17 and 23, solutions to some particular equations were proposed on physical grounds.

In this paper, we present a mathematical approach defining in a consistent way general pseudo-differential operators having a nontrivial set of solutions to the associated nonlocal homogeneous equations. The associated Green functions, as occurs in the case of the local operators, will not be determined *a priori.*

This kind of extension is desirable, as the solutions to the homogeneous equations are the starting point to canonically quantize nonlocal field theories [23, 24] such as those obtained in the context of nonlocal 3D bosonization.

In effective theories *in–in* vacuum expectation values obey equations where the derived nonlocal operators are defined by means of a retarded prescription. Here again the effective homogeneous equations have no nontrivial solutions.

By using our approach, we will write nonlocal effective QED field equations in a way similar to Maxwell's equations, where there is no *a priori* prescription to define the differential operators. In particular, we will show a mode expansion for these equations. The prescription to compute the Green function will be obtained from causality *a posteriori.*

In Section 2, we define the functional spaces in which we build up the framework to describe the pseudo-differential operators. In these spaces we introduce the representation of analytic functionals as "ultradistributions" [25, 26], a convenient way to handle and operate with the usual Green functions of quantum field theories. In Section 3, the relationship between fractional pseudo-differential operators and the fractional derivatives of ref. 8 is exhibited.

In Section 4, we extend our developments to space-time, by defining nonlocal functions of $\Box$. Fractional wave and Klein–Gordon equations are introduced, and solutions for the respective homogeneous equations are given. In Section 5, we apply our method to the nonlocal effective theory of electromagnetism obtained when integrating over the fermionic degrees of freedom [27, 10].

Finally, Section 6 is devoted to a discussion of the developments and results of the paper.

2. FUNCTIONAL SPACES

We will start with the space ζ of entire analytic test functions $\varphi(k)$, rapidly decreasing in any horizontal band. We will call "ultranalytic" any function $\varphi(k) \in \zeta$. They are Fourier transforms of the space $\hat{\zeta}$ of all C^∞ functions $\varphi(x)$ such that $\exp(q|x|D^p\varphi(x))$ is bounded in R for any q and p.

In view of the latter property, $\zeta \supset Z$, where Z is the space of Fourier transforms of K (C^∞ functions on a compact set) (ref. 22, ch. 2, §1.1).

The dual of ζ is the space ζ' of linear functionals defined on ζ. In ζ' we can represent the propagators of a quantum field theory as analytic functionals, with the physical properties that are expected from them [28].

The general form of an analytic functional is

$$\psi(\varphi) = \int_L \psi(k)\varphi(k)\, dk \tag{1}$$

where $\psi(k)$ is an analytic function, and the line L can be deformed as long as no singularity of $\psi(k)$ is crossed. L can be an open line or a closed loop.

Not only L can be deformed without altering $\psi(\varphi)$. Also, the structure of singularities of $\psi(k)$ can be modified. For example, when $\psi(k) = (k-\tau)^{-1}$, the development $(k-\tau)^{-1} = \Sigma_n\, \tau^n k^{-n-1}$ allows the pole at $k=\tau$ to be represented by a series of multipoles located at the origin. This fact is closely related to the expansion of the analytic functional $\delta(k-\tau)$ as a series of $\delta^{(n)}(k)$ (ref. 22, ch. 2, §1.4). We can see then that an analytic functional can be expressed in more than one way.

We are going to use systematically the following representation [25, 26]:

$$\psi(\varphi) = \int_\Gamma dk\; \psi(k)\varphi(k) \qquad (\psi \in \zeta',\ \varphi \in \zeta) \tag{2}$$

where $\psi(k)$ is analytic in $\{k \in C\colon |\text{Im } k| > \rho\}$ and $\psi(k)/k^\rho$ is bounded continuous in $\{k \in C\colon |\text{Im } k| \geq \rho\}$, ρ depending on $\psi(k)$, $\rho \in N$ (N = set of entire numbers).

The path Γ runs from $-\infty$ to $+\infty$ along $\text{Im } k > \rho$ and from $+\infty$ to $-\infty$ along $\text{Im } k < -\rho$.

Note that Eq. (2) tell us that $\psi(k)$ and $\psi(k) + a(k)$, where $a(k)$ is an entire analytic function, represent one and the same functional $\psi(\varphi)$; in particular, the zero functional corresponds to $\psi(k) = a(k)$. In this representation, the unit functional can be given by

$$\psi(k) = \tfrac{1}{2} Sg(\text{Im } k) \tag{3}$$

The natural definition for the Fourier transform of a functional $\psi \in \zeta'$ is given by the functional $\hat{\psi} = \mathcal{F}\psi$ acting over the space of test functions $\hat{\varphi} = \mathcal{F}\varphi \in \hat{\zeta}$ according to

$$\hat{\psi}(\hat{\varphi}) = \mathcal{F}\psi(\mathcal{F}\varphi) = 2\pi\psi(\varphi) \tag{4}$$

For an ultradistribution represented by (2)

$$\begin{aligned}\hat{\psi}(\hat{\varphi}) &= \int_\Gamma dk\; \psi(k) \int_{-\infty}^{+\infty} dx\; \hat{\varphi}(x) e^{ikx} \\ &= \int_{-\infty}^{+\infty} dx\; \hat{\psi}(x)\hat{\varphi}(x)\end{aligned} \tag{5}$$

where

$$\hat{\psi}(x) = \int_{\Gamma} dk\, \psi(k) e^{ikx} \tag{6}$$

The "density" $\hat{\psi}(x)$ is not altered if we add $a(k)$ to $\psi(k)$. The unit functional, Eq. (3), is associated with the density $\hat{\psi}(x) = 2\pi\delta(x)$.

3. PSEUDO-DIFFERENTIAL OPERATORS

A commonly used example of nonlocal differential operator is given by the fractional derivative, as defined in ref. 22 (Ch. 1, §5.5):

$$\frac{d^{\alpha}}{dx^{\alpha}}\, g_{+} = \frac{x_{+}^{-\alpha-1}}{\Gamma(-\alpha)} * g_{+} \tag{7}$$

which is valid over functions g_+ that vanish for $x < 0$. This definition has the property

$$\frac{d^{\beta}}{dx^{\beta}}\left(\frac{d^{\alpha}}{dx^{\alpha}}\, g_{+}\right) = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}}\, g_{+} \tag{8}$$

implying that the unique solution to the homogeneous equation $(d^{\alpha}/dx^{\alpha})h_{+} = 0$ is the trivial one, $h_{+} \equiv 0$ (applying $d^{-\alpha}/dx^{-\alpha}$ to both members). Correspondingly, the equation $(d^{\alpha}/dx^{\alpha})g_{+} = \delta(x)$ leads to a uniquely defined "Green function" $g_{+} = [x_{+}^{\alpha-1}/\Gamma(\alpha)] * \delta(x)$.

Another definition of fractional derivative operator can be given by

$$\frac{d^{\alpha}}{dx^{\alpha}}\, g_{-} = \frac{x_{-}^{-\alpha-1}}{\Gamma(-\alpha)} * g_{-} \tag{9}$$

valid over functions g_- that vanish for $x > 0$. Here again we have a property similar to (8); the Green function is uniquely defined to be $[x_{-}^{\alpha-1}/\Gamma(\alpha)] * \delta(x)$. We stress that these fractional derivatives have no nontrivial solution to the associated homogeneous equations.

On the other hand, now we will define general pseudo-differential operators, having a nontrivial set of solutions to the associated nonlocal homogeneous equations, and (as a consequence) the associated Green functions, as occurs in the case of the local operators, will not be determined *a priori*. This kind of extension provides a formal base for nonlocal 3D bosonization as a theory in Minkowski space; also, we will be able to interpret nonlocal effective field equations in a similar way to the local case (see Section 5).

In order to work properly with some nonlocal pseudo-differential operators we are going to introduce the following operation on the functionals $\hat{\psi}$ in (2).

Let us consider a function $f(k)$ such that $f(k)$ is analytic in $\{k \in C/|\text{Im } k| > \beta\}$ and $f(k)/k^{\beta}$ is bounded continuous in $\{k \in C/|\text{Im } k| \geq \beta\}$, β depends on $f(k)$, $\beta \in N$.

Then we define

$$f\hat{\psi}(\hat{\varphi}) = \int_{-\infty}^{+\infty} \left[f\left(-i\frac{d}{dx}\right)\hat{\psi}(x) \right]\hat{\varphi}(x) \tag{10}$$

$$f\left(-i\frac{d}{dx}\right)\hat{\psi}(x) = \int_{\Gamma} dk\, f(k)\psi(k)e^{ikx} \tag{11}$$

where the path Γ runs from $-\infty$ to $+\infty$ along $\text{Im } k > \beta + \rho$ and from $+\infty$ to $-\infty$ along $\text{Im } k < -\beta - \rho$.

We know that the functional $\hat{\psi}(\hat{\varphi})$ does not change when we add an arbitrary entire function $a(k)$ to $\psi(k)$. However, such an addition in Eq. (11) gives rise to a new term

$$A(x) = \int_{\Gamma} dk\, f(k)a(k)e^{ikx} \tag{12}$$

When $f(z)$ is an entire function (for example, in the case of polynomial functions of the derivative operator), $A(x) \equiv 0$.

If $f(z) = z^{-1}$, $f(-i\, d/dx)$ is the inverse of the derivative (an integration). In this case, Eq. (11) gives a primitive of $\hat{\psi}(x)$; using Cauchy's theorem, the additional term (12) is $A(x) = -2\pi i a(0)$. Of course, an integration should give a primitive plus an arbitrary constant.

Analogously, a double (iterated) integration with $f(z) = z^{-2}$ gives a primitive plus

$$A(x) = \int_{L} dk\, \frac{a(k)}{k^2}\, e^{ikx} = \gamma + \delta x$$

where γ and δ are arbitrary constants. It is understandable that a more complex structure of singularities of $f(z)$ gives rise to a more complicated $A(x)$.

To solve some pseudo-differential equations, we can work directly with the functions $\psi(k)$ representing the analytic functional ψ in Eq. (2), the null functional being represented by an arbitrary entire function $a(k)$.

For a solution to the homogeneous equation

$$f\left(-i\frac{d}{dx}\right)\hat{\psi}(x) = 0 \tag{13}$$

we write

$$f(k)\psi(k) = a(k)$$

that is,

$$\psi(k) = f^{-1}(k)a(k) \tag{14}$$

Then replacing in Eq. (6), we obtain

$$\hat{\psi}(x) = \int_\Gamma dk\, f^{-1}(k)a(k)e^{ikx} \tag{15}$$

In the case where the singularities of f^{-1} are concentrated on the real axis, the path Γ can be deformed to get

$$\hat{\psi}(x) = \int_{-\infty}^{+\infty} dk\, [f^{-1}(k + i0) - f^{-1}(k - i0)]a(k)e^{ikx} \tag{16}$$

To solve the inhomogeneous equation

$$f\left(-i\frac{d}{dx}\right)\hat{\psi}(x) = \hat{\chi}(x) \tag{17}$$

where $\hat{\chi}(x)$ is a given well-behaved function, we write

$$f(k)\psi(k) = \chi(k) + a(k)$$

$$\psi(k) = f^{-1}(k)\chi(k) + f^{-1}(k)a(k) \tag{18}$$

where $\hat{\chi}(x) = \int_\Gamma dk\, (\chi(k) + a(k)) \exp ikx = \int_\Gamma dk\, \chi(k) \exp ikx$. The last term in Eq. (18) can be recognized as a general solution to the homogeneous equation (13).

Now, we would like to comment on the composition properties of the pseudo-differential operators we defined. Suppose we want to solve Eq. (13). If we apply to both members the operator $g(-i\, d/dx)$, we obtain

$$g\left(-i\frac{d}{dx}\right)\left[f\left(-i\frac{d}{dx}\right)\hat{\psi}\right] = 0 \tag{19}$$

The square bracket is a solution to the homogeneous equation for g. Accordingly,

$$f\left(-i\frac{d}{dx}\right)\hat{\psi}(x) = \int_\Gamma dk\, g^{-1}a(k)e^{ikx}$$

Now we have an inhomogeneous equation for $\hat{\psi}$, whose solution is

$$\hat{\psi}(x) = \int_\Gamma dk\, f^{-1}(k) g^{-1}(k) a(k) e^{ikx} + \int_\Gamma dk\, f^{-1}(k) b(k) e^{ikx} \tag{20}$$

The second term is a general solution to Eq. (13); of course, the set of solutions to Eq. (19) contains the space of solutions to Eq. (13). On the other hand, if we define the composition of two operators associated with $f(k)$ and $g(k)$ to be an operator $g(-i\, d/dx) \circ f(-i\, d/dx)$ associated with $f(k)g(k)$, the solutions to the equation

$$\left[g\left(-i\frac{d}{dx}\right) \circ f\left(-i\frac{d}{dx}\right) \right] \psi = 0 \tag{21}$$

are given by

$$\int_\Gamma dk\, f^{-1}(k) g^{-1}(k) a(k) e^{ikx} \tag{22}$$

Then, we see that only in the case where $g(-i\, d/dx)$ is associated with an entire function $g(k)$ [for example, in the case of a local operator $g(-i\, d/dx)$] can we choose the entire function $a(k)$ to be $g(k)$ times an entire function, to conclude that the space of solutions to (21) contains the space of solutions to (13). This is in contrast with the properties (8) of the fractional derivatives, which lead to a trivial space of solutions to the associated homogeneous equations.

Finally, to see the connection with the fractional derivative, we can apply the pseudo-differential operator $(-i\, d/dx)^\alpha$ to a function g_+ (g_-) that vanishes for $x < 0$ ($x > 0$). According to (6), the representing function $\psi_+(k)$ [$\psi_-(k)$] is zero for Im $k > 0$ (Im $k < 0$). Then, we have

$$g_+(x) = \int_\Gamma dk\, \psi_+(k) e^{ikx} = -\int_{\mathrm{Im}\, k < -\rho} dk\, \psi_+(k) e^{ikx} = -\int_R dk\, \psi_+(k - i0) e^{ikx} \tag{23}$$

and

$$g_-(x) = \int_\Gamma dk\, \psi_-(k) e^{ikx} = \int_{\mathrm{Im}\, k > \rho} dk\, \psi_-(k) e^{ikx} = \int_R dk\, \psi_-(k + i0) e^{ikx} \tag{24}$$

Therefore

$$\left(-i\frac{d}{dx}\right)^{\alpha} g_{\pm} = \int_R dk\, k^{\alpha}\psi_{\pm}(k)e^{ikx}$$
$$= \mp\int_R dk\,(k \mp i0)^{\alpha}\psi_{\pm}(k \mp i0)e^{ikx}$$
$$= \mp\mathcal{F}(k \mp i0)^{-\alpha} * g_{\pm} = \mp 2\pi e^{\mp i(\pi/2)\alpha}\frac{x_{\pm}^{-\alpha-1}}{\Gamma(-\alpha)} * g_{\pm} \quad (25)$$

Up to a factor, Eq. (25) contains Gelfand's fractional derivative definitions (7) and (9) which are valid over functions g_+, g_- that vanish for $x < 0$, $x > 0$, respectively.

4. LORENTZ INVARIANT PSEUDO-DIFFERENTIAL OPERATORS

In this section we examine some nonlocal functions $f(\Box)$ of $\Box = \partial_0^2 - \bar{\partial}^2$, where $\bar{\partial}^2$ is the $(d-1)$-dimensional Laplacian operator.

The ultradistributions, Eq. (2), depend now on a $(d-1)$-dimensional vector $\bar{k}$ as parameter $\psi(k_0) \to \psi(k_0, \bar{k})$.

The Fourier transform of Eq. (6) gives

$$\hat{\psi}(x_0, \bar{k}) = \int_\Gamma dk_0\, \psi(k_0, \bar{k})e^{ik_0x_0} \quad (26)$$

With the usual Fourier transform in the space of the parameters $\bar{k}$, we obtain

$$\hat{\psi}(x) = \hat{\psi}(x_0, \bar{x}) = \int_{-\infty}^{+\infty} d\bar{k}\, \hat{\psi}(x_0, \bar{k})e^{-i\bar{k}\bar{x}} \quad (27)$$

We define the operation $f(\Box)$ on $\hat{\psi}(x)$ by

$$f(\Box)\hat{\psi}(x) = \int d\bar{k}\, f(\partial_0^2 + \bar{k}^2)\hat{\psi}(x_0, \bar{k})e^{-i\bar{k}\bar{x}} \quad (28)$$

and [cf. Eq. (11)]

$$f(\partial_0^2 + \bar{k}^2)\hat{\psi}(x_0, \bar{k}) = \int_\Gamma dk_0\, f(-k_0^2 + \bar{k}^2)\psi(k_0, \bar{k})e^{+ik_0x_0} \quad (29)$$

We can now solve the homogeneous (d-dimensional) equation

$$f(\Box)\hat{\psi}(x) = 0 \quad (30)$$

According to (28), Eq. (30) implies

$$f(\partial_0^2 + \bar{k}^2)\hat{\psi}(x_0, \bar{k}) = 0$$

Due to (29), we have for the representing functions

$$\psi(k_0, \bar{k}) = f^{-1}(-k_0^2 + \bar{k}^2)a(k_0, \bar{k}) \tag{31}$$

where $a(k_0, \bar{k})$ is an arbitrary entire function of k_0, for any value of $\bar{k}$.

From (26) and (31) we get

$$\hat{\psi}(x_0, \bar{k}) = \int_\Gamma dk_0\, f^{-1}(-k_0^2 + \bar{k}^2)a(k_0, \bar{k})e^{ik_0x_0} \tag{32}$$

If the singularities of $f^{-1}(-k_0^2 + \bar{k}^2)$ (cuts and poles) lie on the real k_0-axis, the integration in (32) can then be taken along the real axis, yielding

$$\hat{\psi}(x_0, \bar{x}) = \int d\bar{k} \int_{-\infty}^{+\infty} dk_0\, \Delta(f^{-1})a(k_0, \bar{k})e^{i(k_0x_0 - \bar{k}\cdot\bar{x})} \tag{33}$$

where Δ is the discontinuity at the singularities:

$$\begin{aligned}\Delta(f^{-1}) &= f^{-1}(-(k_0 + i0)^2 + \bar{k}^2) - f^{-1}(-(k_0 - i0)^2 + \bar{k}^2)\\ &= 2iSgk_0\sigma(k^2)\end{aligned} \tag{34}$$

Here, we have also defined $\sigma(k^2)$, the weight function associated with the k^2-mode.

For $f(z) = (z + m^2)^\alpha$, determined by a cut along the negative real z-axis, running from $-\infty$ to $-m^2$, the analytic function of k_0, $f^{-1} = (-k_0^2 + \bar{k}^2 + m^2)^{-\alpha}$, presents a cut along the real k_0-axis running from $-\infty$ to $k_0 = -\omega$ and another twin cut from $k_0 = +\omega$ to $+\infty$ ($\omega = +\sqrt{\bar{k}^2 + m^2}$). In this case, the discontinuity function (34) is given by

$$\begin{aligned}\Delta(f^{-1}) &= [-k_0^2 + \omega^2 - i0Sg(k_0)]^{-\alpha} - [-k_0^2 + \omega^2 + i0Sg(k_0)]^{-\alpha}\\ &= Sg(k_0)[(-k_0^2 + \omega^2 - i0)^{-\alpha} - (-k_0^2 + \omega^2 + i0)^{-\alpha}]\end{aligned}$$

Using

$$\begin{aligned}(k + i0)^{-\alpha} &= k_+^{-\alpha} + e^{-ik\alpha}k_-^{-\alpha}\\ (k - i0)^{-\alpha} &= k_+^{-\alpha} + e^{ik\alpha}k_-^{-\alpha}\end{aligned} \tag{35}$$

we get

$$\Delta(f^{-1}) = 2i\sin(\pi\alpha)Sgk_0(-k_0^2 + \omega^2)_-^{-\alpha} = 2i\sin(\pi\alpha)Sgk_0(k_0^2 - \omega^2)_+^{-\alpha} \tag{36}$$

For $\alpha = 1$, Eq. (30) is Klein–Gordon equation, the distribution $(k_0^2 - \omega^2)_+^{-\alpha}$ has a pole (ref. 22, Ch. 3, §3.4) while $\sin(\pi\alpha)$ is zero; therefore, in this case, the weight function (36) comes from the residue of $(k_0^2 - \omega^2)_+^{-\alpha}$ at $\alpha = 1$, that is, $\delta(k_0^2 - \omega^2) = \delta(k^2 - m^2)$. Then, for $\alpha \to 1$, Eq. (36) gives $\Delta \to \text{const} \cdot \delta(k^2 - m^2)$, which is the well-known invariant free wave solution. For a fractional α, the weight function gives a continuum of modes with $k^2 \geq m^2$. In this sense we can say that (36) represents modes corresponding to a continuum of masses $k^2 = \mu^2 \geq m^2$. the free wave $\delta(k^2 - m^2)$ concentrated on $k^2 = m^2$ changes for fractional α into $(k^2 - m^2)_+^{-\alpha}$, which is spread from $k^2 = m^2$ to $k^2 \to \infty$ (see also ref. 23).

Note that, according to Section 3, in the case of the nonlocal equation $\Box^{1/2}\hat{\psi} = 0$, for example, it is not possible to conclude (applying $\Box^{1/2}$) that $\hat{\psi}$ must be a solution to $\Box\hat{\psi} = 0$, implying just zero mass modes; instead, the solution contains a continuum of massive modes, with weight $\sim (k_+^2)^{-1/2}$.

Let us now find Green functions for the fractional Klein–Gordon equation:

$$(\Box + m^2)^\alpha \hat{G}(x) = \delta(x) \tag{37}$$

As the pseudo-differential wave equation $(\Box + m^2)^\alpha \hat{\psi}(x) = 0$ has nontrivial solutions, this Green function is not uniquely determined, as occurs in the case of the local operators. This is in contrast to the usual approach to fractional wave equations where the solution to the homogeneous equation is trivial and the Green function is uniquely determined.

A particular solution to (37) is

$$\hat{G}(x_0, \bar{\bar{k}}) = \frac{1}{2}\int dk\,[(-k^2 + m^2 - i0)^{-\alpha} + (-k^2 + m^2 + i0)^{-\alpha}]e^{ikx} \tag{38}$$

This is the principal value (or "Wheeler") Green function for the fractional Klein–Gordon equation.

The two terms in the square bracket of (38) are, respectively, the causal and anticausal Green functions for the fractional pseudo-differential wave equation. For $\alpha = 1$ they are the usual Feynman propagator and its complex conjugate. The Fourier transforms of those two terms can be found in ref. 22, Ch. 3, §2.6.

Other Green functions can be found by adding to (38) solutions to the homogeneous equation. We can add together (33) (with $a = \pm 1/2$) and (38) to get

$$\hat{G}_\pm(x_0, \bar{\bar{x}}) = \int dk\,[(k^2 - m^2)_-^{-\alpha} + e^{\pm i\pi\alpha Sgk_0}(k^2 - m^2)_+^{-\alpha}]e^{ikx} \tag{39}$$

When $m = 0$, the two terms of (38) and the two Green functions of (39) coincide with the four Lorentz-invariant propagators for the fractional wave equation found in ref. 15; the Feynman (causal) function coincides with the propagator used in ref. 7 to regularize the matrix elements of quantum electrodynamics.

5. SOLUTIONS TO THE HOMOGENEOUS EQUATIONS IN EFFECTIVE QED

Now we discuss the nonlocal field equations we have defined in the context of effective electromagnetism.

The usual effective action, obtained by integration of the (quantum) fermionic degrees of freedom, when written in Minkowski space leads to effective (*in–out*) field equations which are neither real nor causal. On the other hand, a "close time path" formalism can be considered that produces (real and causal) effective field equations for the expectation values of the hermitian Heisenberg field.

For example, this formalism leads to the equation for the *in–in* vacuum expectation values of the electromagnetic field [27, 10]:

$$H(\Box)\partial_\mu F^{\mu\nu} = J^\nu_{clas}, \qquad H(\Box) = \left(1 - \frac{e^2}{\pi^2} F(\Box)\right) \tag{40}$$

where $F(\Box)$ comes from the one-loop vacuum polarization tensor,

$$F(\Box) = \frac{1}{8}\int_0^1 dt\,(1 - t^2)\ln\left[\frac{m^2 + 1/4(1 - t^2)\Box}{\mu^2}\right] \tag{41}$$

and the nonlocal operator in (40) is defined by means of a retarded prescription, that is, when evaluating $H(\Box))\partial_\mu F^{\mu\nu}$ in k-space, the cut of $F(-k^2)$, must be avoided by means of a retarded path. j_{clas} is a classical charged source.

Note that the kinetic operator (40) *itself* and not just its inverse is defined by means of a retarded prescription, which leads to a uniquely defined (classical) causal relation between the classical source and the *in–in* vacuum expectation value of the electromagnetic field; this relationship comes from the (quantum) causal Feynman propagator defining QED.

Now, we would like to consider the (nonlocal) effective equation (40) (which includes quantum fermion corrections) in a similar way to the classical (local) electromagnetic equation, where the retarded prescription is not included *a priori* in the kinetic operator (but is imposed *a posteriori* by means of a causality argument). In particular, such an equation could also be quantized.

Then, instead of considering these equations with an operator defined by a retarded prescription, we could consider them in the sense defined in

(28) and (29), where the nonlocal operator has a closer analogy with the local case, accepting a nontrivial set of solutions to the homogeneous effective equations.

Using our definition for $H(\Box)$, we can solve (in the Lorentz gauge) the homogeneous effective equation

$$H(\Box)\Box A^{\mu} = 0 \tag{42}$$

From (33) we obtain

$$A^{\mu}(x) = \int d\vec{k} \int_{-\infty}^{+\infty} dk_0\, \Delta((-k^2 H)^{-1}) a^{\mu}(k_0, \vec{k}) e^{ikx} \tag{43}$$

where $\overline{a}^{\mu}(k_0, \vec{k}) = a^{\mu}(-k_0, -\vec{k})$ for A^{μ} to be a real field, and we have the gauge condition, $k \cdot a(k) = 0$.

The discontinuity function is

$$\begin{aligned}
&\Delta\left(\frac{1}{-k^2 H(-k^2)}\right) \\
&\quad = i\pi Sgk_0 \delta(k^2)\left[\frac{1}{H(-k^2 + i0)} + \frac{1}{H(-k^2 - i0)}\right] \\
&\qquad + Sgk_0 P\left(\frac{1}{k^2}\right)\left[\frac{1}{H(-k^2 + i0)} - \frac{1}{H(-k^2 - i0)}\right] \\
&\quad = 2iSgk_0\left[\frac{\pi}{\mathcal{R}(0)}\delta(k^2) + \frac{1}{k^2}\frac{\mathcal{I}}{\mathcal{R}^2 + \mathcal{I}^2}\right] \\
&\mathcal{R}(0) = 1 - \frac{e^2}{12\pi^2}\ln\frac{m^2}{\mu^2}
\end{aligned} \tag{44}$$

where $\mathcal{R}$ and $\mathcal{I}$ are the real and imaginary parts of $H(-k^2 - i0)$, respectively, and we used the fact that [because of the unitarity cut of $F(-k^2)$] the imaginary part of $H(-k^2 - i0)$ is zero for $k^2 < 4m^2$.

Note that the field modes in the continuum, with mass parameter $\mu^2 \geq 4m^2$, coming from the second term in (44) cannot be associated with asymptotic free wave solutions representing a particle with mass μ^2, that is, there is no solution such as exp ikx, $k^2 = \mu^2$; the function $a^{\mu}(k_0, \vec{k})$ in (43) cannot be a Dirac delta, as it must be a function analytic in k_0. On the other hand, the $\delta(k^2)$-function in the first term gives the usual free wave solutions exp ikx, $k^2 = 0$, representing the photon.

Note also that the weight function [bracket in (44)] is positive definite, as the imaginary part

$$\mathcal{I} = e^2/8\pi \int_0^1 dt(1 - t^2)\theta((1 - t^2)k^2 - 4m^2)$$

is positive; it is related (via the optical theorem) with the total cross section to produce a fermion pair (which is of course positive). This is in contrast with the case of a higher order field equation, where the weight function is not positive definite. For instance, when $f(\Box) = (\Box + m_1^2)(\Box + m_2^2)$, the weight function is $1/(m_2^2 - m_1^2)[\delta(k^2 - m_1^2) - \delta(k^2 - m_2^2)]$. This indefiniteness finally leads to an indefinite conserved energy, that is, the instability problem associated with higher order field equations [29]. Then, unlike a higher order equation, the nonlocal effective equation (42) does not present instability problems (as expected), as the underlying (local) theory is physically well defined.

Finally, we could quantize the classical system represented by equation (42). This can be done by imposing Heisenberg's equation on the on-shell field (43) to obtain the a-commutators and by imposing the gauge condition on the physical modes $\partial \cdot A^- |phys\rangle = 0$, where A^- is the annihilation part of the electromagnetic field [23].

The obtained propagator, computed as the vacuum expectation value of the T-product for two nonlocal fields, is

$$i\eta_{\mu\nu} \int_{\Gamma_F} dk \frac{e^{ikx}}{(-k^2 H(k^2))} \tag{45}$$

that is, the usual electromagnetic Feynman's propagator, including the virtual fermion self-energy.

6. DISCUSSION

Nonlocal kinetic operators are currently used in many problems of theoretical physics. A general Lorentz invariant kinetic operator is given by a function $f(\Box)$, where $f(-k^2)$ is an analytic function of k_0, with a given set of singularities (cuts).

Usually, in order to define these operators, a prescription to avoid these singularities is considered in an analogous way to Gelfand's fractional derivative definition [22]. In that context, given a nonlocal operator, the associated nonlocal homogeneous equation has no nontrivial solutions, and therefore the associated Green function is uniquely determined.

In this paper, we presented a different mathematical approach, defining general nonlocal pseudo-differential operators acting on a space of ultradistributions. This is a natural extension, implying a nontrivial set of solutions to the associated nonlocal homogeneous equations.

Our method is based on defining the space ζ of ultraanalytic functions (see Section 2). Its dual space ζ' contains the propagators appearing in perturbative quantum field theories. It is also flexible enough to allow for them a representation in terms of ultradistributions which is both simple and general.

The Fourier-transformed species $\hat{\zeta}$ and $\hat{\zeta}'$ translate the functions of ζ and ζ' into functions of coordinates where the pseudo-differential operators are supposed to act. In this way we can define nonlocal operations on (generalized) functions of the coordinates.

We have shown the connection with the operators $\Box^{\alpha}$ defined and discussed in ref. 15 and with the fractional derivative of ref 22. We also exhibited the corresponding solutions to the homogeneous equations. As a consequence, we are able to show that the free solutions to the wave equation or the Klein–Gordon equation, with their sharp masses, are "spread" when the equations are fractional into a superposition of a continuum of massive modes with support in the interior of the light cone.

Furthermore, the causal propagators for the fractional equations are seen to coincide with the analytically regularized propagators introduced in ref. 7. It is then possible to interpret this regularization method as the matrix elements one would write for a nonlocal theory having no ultraviolet divergences. The usual infinities appear as poles for the local limit $\alpha \to 1$.

We also analyzed, in the frame of our extended approach, the (nonlocal) effective equation of electromagnetism obtained when integrating over the fermionic degrees of freedom. The general solution to the homogeneous (effective) Maxwell equation contains a zero mass field, which represents the photon, and an additional term representing a continuum of massive modes. The modes in the continuum cannot be associated with "sharp mass" free waves (see also ref. 24) representing massive particles. The quantization of the nonlocal effective field leads to Feynman's photon propagator including the quantum corrections due to virtual fermions.

It is interesting to note that, unlike the higher derivative case, the weight function for the nonlocal effective theory is positive definite. Then, the theory does not present instability problems, as expected, as the underlying local theory is physically well defined.

We see than that with the chosen procedures we have convenient tools with which we can handle different pseudo-differential nonlocal problems. The scheme also provides a formal base for the quantization of nonlocal field theories [23], as those obtained in the context of bosonization in higher dimensions.

ACKNOWLEDGMENTS

This work was partially supported by Centro Latinoamericano de Física (CLAF), Conselho Nacional de Desenvolvimento Científico e Tecnológico

(CNPq, Brasil), Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET, Argentina), and Comisión de Investigaciones Científicas de la Pcia. de Bs. As. (CIC, Argentina).

REFERENCES

[1] J. A. Wheeler and R. P. Feynman, *Rev. Mod. Phys.* **17** (1945), 157.
[2] M. N. Volkov and G. V. Efimov, *Sov. Phys. Usp.* **23** (1980), 94.
[3] N. V. Krasnokov, *Theor. Math. Phys.* **73** (1987), 1184.
[4] D. A. Eliezer and R. P. Woodard, *Nucl. Phys. B* **325** (1984), 389.
[5] A. Hata, *Nucl. Phys. B* **329** (1990), 698.
[6] A. Pais and G. E. Uhlembeck, *Phys. Rev.* **79** (1950), 145.
[7] C. G. Bollini, J. J. Giambiagi, and A. Gonzáles Domíngues, *Nuovo Cimento* **31** (1964), 550.
[8] A. O. Barvinsky and C. A. Vilkovisky, *Nucl. Phys. B* **282** (1987), 163.
[9] A. O. Barvinsky and C. A. Vilkovisky, *Nucl. Phys. B* **333** (1990), 471.
[10] D. A. R. Dalvit and F. D. Mazzitelli, *Phys. Rev. D* **50** (1994), 1002.
[11] E. C. Marino, *Phys. Lett. B* **263** (1991), 63.
[12] D. G. Barci, C. D. Fosco, and L. E. Oxman, *Phys. Lett. B* **375** (1996), 267.
[13] J. C. LeGuillou, C. Núñez, and F. Schaposnik, *Ann. Phys. (NY)* **251** (1996), 426.
[14] R. Banerjee and E. C. Marino, *Phys. Rev. D* **56** (1997), 3763.
[15] C. G. Bollini and J. J. Giambiagi, *J. Math. Phys.* **34** (1993), 610.
[16] J. J. Giambiagi, Huygens principle in $2N + 1$ dimensions for non-local operators $\Box^{\alpha}$, preprint CBPF-NF-025/91-Brasil.
[17] R. L. P. G. Amaral and E. C. Marino, *J. Phys. A* **25** (1992), 5183.
[18] E. C. Marino, *Nucl. Phys. B* **408**[FS] (1993), 551.
[19] G. M. Zaslavsky, *Physica D* **76** (1994), 110.
[20] W. Wyss, *J. Math. Phys.* **27** (1986), 2782.
[21] W. R. Schneider and W. Wyss, *J. Math. Phys.* **30** (1989), 134.
[22] I. M. Gelfand and G. E. Shilov, *Les Distributions,* Vol. I, Dunod, Paris (1972).
[23] D. G. Barci, L. E. Oxman, and M. C. Rocca, *Int. J. Mod. Phys. A* **11** (1996), 2111.
[24] D. G. Barci and L. E. Oxman, *Mod. Phys. Lett. A* **12** (1997), 493.
[25] L. Sebastião e Silva, *Math. Ann.* **136** (1958), 58.
[26] M. Hasuni, *Tôhoku Math. J.* **13** (1961), 94.
[27] R. D. Jordan, *Phys. Rev. D* **36** (1987), 3604.
[28] C. G. Bollini, L. E. Oxman, and M. Rocca, *J. Math. Phys.* **35** (1994), 4429.
[29] S. W. Hawking, In *Quantum Field Theory and Quantum Statistics,* Vol. 2, I. A. Batalin, C. J. Isham, and G. A. Vilkovisky, eds., Adam Hilgar, Bristol (1987), p. 129.